



Block Toeplitz Operators on Vector-Valued Doubling Fock Spaces

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Article's Information	Abstract
Received: 27.08.2025 Accepted: 04.12.2025 Published: 15.06.2026 Keywords: Block Toeplitz operators, Vector-valued doubling, Fock spaces, Matrix-valued symbol, Fredholm properties. http://doi.org/10.22401/ANJS.29.2.14 *Corresponding author: aamena.raimmohammed@nahrainuniv.edu.iq	This paper intends to examine the identity of the block Toeplitz operator BT_Y^m, with a matrix-valued symbol Y, on vector-valued doubling Fock spaces $F_{\vartheta,m}^p$. For an integer m, the spaces $F_{\vartheta,m}^p$ are characterized as the spaces of all vector functions F whose component functions belong to the doubling Fock space F_{ϑ}^p, and the operator BT_Y^m is defined upon the $F_{\vartheta,m}^p$. However, fundamental properties like boundedness, compactness of BT_Y^m are explored. Additionally, for a specific type of matrix-valued symbol Y, the necessary and sufficient condition on the symbol Y that makes BT_Y^m Fredholm on $F_{\vartheta,m}^p$ are established.



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1. Introduction

The study of Toeplitz operator T_{η} , defined on Bergman, Hardy and Fock spaces, with wide classes of functions η , has a considerable history. The importance of these operators comes from the massive problems in physics and the field of quantum mechanics that are connected with properties and the spectral properties of Toeplitz operators, an example, of which the boundedness and compactness of T_{η} play a natural role with the heat flow $\tilde{\eta}^p$ of η , [1]. The block Toeplitz operators arise in the study of the classical dimer model on the triangular lattice, in particular. The approximate calculation of determinants of these types of operators, have a significant role in the study of high-temperature super conductivity, [2]. In [3], Böttcher and Silbermann define the block Toeplitz operator on the space $l_N^p(C)$, and they investigate the boundedness, the compactness and the Fredholm properties of these types of operators. In [4], Gu. Hendricks and Rutherford postulate that when the block Toeplitz operators are hyponormal on the space of all vector functions where the component functions are in Hardy spaces. Another research about the hyponormality and also subnormality of block Toeplitz operators defined on vector-valued Hardy space due to the authors in [5]. Also, on vector-valued Hardy spaces, I. Hwang and W. Lee in [6], investigated BT_Y^m with a matrix-valued rational

symbol. For a vector-valued Bergman space, J. Lee in [7], inspected the hyponormality of BT_Y^m with matrix-valued polynomial functions of degree 2. Moreover, in [8], block Toeplitz operators are defined on vector-valued Fock space $F_{\vartheta,m}^p$, for a special case, when $\vartheta = \frac{1}{2}|\omega|^2$.

2. Preliminaries

The aim of this section is to reestablish certain known definitions that are needed in subsequent sections.

Definition 2.1, [9].

Hypothesize that $0 \neq \vartheta$ is a subharmonic function on C such that $d\nu = \Delta\vartheta dR$ is a doubling measure, where $\Delta\vartheta = \vartheta_{xx} + \vartheta_{yy}$ and dR is the Lebesgue area measure on C . For $1 < p < \infty$, L_{ϑ}^p is the space of all Lebesgue measurable functions φ defined on C with

$$\|\varphi\|_{p,\vartheta}^p = \int_C |\varphi(\omega)|^p e^{-p\vartheta(\omega)} dR(\omega) < \infty.$$

The doubling Fock space F_{ϑ}^p is defined as follows

$$F_{\vartheta}^p = L_{\vartheta}^p \cap H(C),$$

where, $H(C)$ is the set of all holomorphic functions defined on C . It is a closed subspace of L_{ϑ}^p . For $p \geq 1$,

it is known that the doubling Fock space F_{ϑ}^p is a Banach space, and for $p = 2$, F_{ϑ}^2 is considered to be a Hilbert space with the next inner product:

$$\langle \varphi_1, \varphi_2 \rangle = \int_C \varphi_1(\omega) \overline{\varphi_2(\omega)} e^{-2\vartheta(\omega)} dR(\omega).$$

Definition 2.2, [10].

The Toeplitz operator $T_{\eta}: F_{\vartheta}^2 \rightarrow F_{\vartheta}^2$ and the Hankel operator $H_{\eta}: F_{\vartheta}^2 \rightarrow (F_{\vartheta}^2)^{\perp}$ with the symbol $\eta \in L^{\infty}(C)$ are defined by

$$T_{\eta}(\varphi) = P_{\vartheta}(\eta\varphi) \quad \text{and}$$

$$H_{\eta}(\varphi) = (I - P_{\vartheta})(\eta\varphi) = \eta\varphi - P_{\vartheta}(\eta\varphi)$$

for all $\varphi \in F_{\vartheta}^2$, where $P_{\vartheta}: L_{\vartheta}^2 \rightarrow F_{\vartheta}^2$ is the orthogonal projection and I is the identity operator on F_{ϑ}^2 . It is known that the Toeplitz operator T_{η} is bounded and $\|T_{\eta}\| = \|\eta\|_{\infty}$. The relation between Toeplitz and Hankel operators can be summarized by the following equation

$$T_{\eta\lambda} = T_{\eta}T_{\lambda} + T_{\eta}H_{\lambda}, \quad \text{for all } \eta, \lambda \in L^{\infty}(C) \quad \dots (2.1)$$

We end this section by recalling the definition of Fredholm operators.

Definition 2.3, [11].

The condition for a bounded linear operator T on a Banach space $\mathcal{E}$ to be Fredholm is when the dimension of the kernel T is finite and the dimension of $(\mathcal{E} / T(\mathcal{E}))$ is not infinite. The index equation of T is given by

$$\text{ind}(T) = \dim(\text{Ker } T) - \dim(\mathcal{E} / T(\mathcal{E})).$$

If K is a compact operator on $\mathcal{E}$, then $T + K$ is Fredholm and $\text{ind}(T + K) = \text{ind}(T)$.

3. Block Toeplitz on vector-valued doubling Fock spaces

This section is devoted to defining of the space $F_{\vartheta,m}^p$ and providing the definition of the operator $BT_Y^m: F_{\vartheta,m}^p \rightarrow F_{\vartheta,m}^p$.

Definition 3.1

Hypothesize that $m \in N$, $1 < p < \infty$, and $0 = \vartheta$ is a subharmonic function on C . The space $L_{\vartheta,m}^p$ is defined as a collection of all vector functions $F = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}$, where the component functions $f_1, \dots, f_m$, are in L_{ϑ}^p , with the norm

$$\|F\|_{\vartheta,m} = \sum_{l=1}^m \|f_l\|_{p,\vartheta}^p \quad \dots (3.1)$$

The vector-valued doubling Fock space $F_{\vartheta,m}^p$ is the family of all vectors $F = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{pmatrix}$, where the component functions $\varphi_1, \dots, \varphi_m$, are in F_{ϑ}^p , it is a closed subspace of $L_{\vartheta,m}^p$.

Definition 3.2

Hypothesize that $m \in N$, the symbol $L_m^{\infty}(C)$ is used to indicate the space of all square matrices Y , with the inlets η_{ij} are functions in $L^{\infty}(C)$, $1 \leq i, j \leq m$. Y is called a matrix-valued function and

$$\|Y\|_{L_m^{\infty}(C)} = \sum_{i,j=1}^m \|\eta_{i,j}\|_{L^{\infty}(C)}.$$

Given $Y \in L_m^{\infty}(C)$ and for any $F \in F_{\vartheta,m}^p$, we set the formula of the block Toeplitz operator BT_Y^m on $F_{\vartheta,m}^p$ as follows:

$$\begin{aligned} BT_Y^m F &= BP_{\vartheta,m}(YF) = BP_{\vartheta,m} \left(\sum_{j=1}^m \eta_{ij} \varphi_j \right)_{1 \leq i \leq m}^T \\ &= \left(\sum_{j=1}^m T_{\eta_{ij}} \varphi_j \right)_{1 \leq i \leq m}^T \quad \dots (3.2) \\ &= \begin{pmatrix} T_{\eta_{11}} & \cdots & T_{\eta_{1m}} \\ \vdots & \ddots & \vdots \\ T_{\eta_{m1}} & \cdots & T_{\eta_{mm}} \end{pmatrix}_{m \times m} \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{pmatrix}_{m \times 1} \end{aligned}$$

where $BP_{\vartheta,m}: L_{\vartheta,m}^p \rightarrow F_{\vartheta,m}^p$ is the projection operator. The next result shows that BT_Y^m is bounded.

Lemma 3.3

Hypothesize that $m \in N$. Then BT_Y^m is bounded and linear on $F_{\vartheta,m}^p$.

Proof:

Suppose that $Y \in L_m^{\infty}(C)$, then

$$Y = \begin{pmatrix} \eta_{11} & \cdots & \eta_{1m} \\ \vdots & \ddots & \vdots \\ \eta_{m1} & \cdots & \eta_{mm} \end{pmatrix}_{m \times m}$$

where, $\eta_{ij} \in L^{\infty}(C)$. This indicates that $T_{\eta_{ij}}$ is bounded on F_{ϑ}^p for all $i, j = 1, \dots, m$. Using Equation (3.2),

$$\|BT_Y^m(F)\|_{\vartheta,m} = \left\| \begin{pmatrix} T_{\eta_{11}}\varphi_1 + \cdots + T_{\eta_{1m}}\varphi_m \\ \vdots \\ T_{\eta_{m1}}\varphi_1 + \cdots + T_{\eta_{mm}}\varphi_m \end{pmatrix}_{m \times 1} \right\|_{\vartheta,m} = \left(\{T_{\eta_{11}}\varphi_{1\kappa}\}_{\kappa=1}^\infty + \cdots \{T_{\eta_{1m}}\varphi_{1\kappa}\}_{\kappa=1}^\infty, \dots, \left(\{T_{\eta_{m1}}\varphi_{1\kappa}\}_{\kappa=1}^\infty + \cdots \{T_{\eta_{mm}}\varphi_{1\kappa}\}_{\kappa=1}^\infty \right) \right)$$

From (3.1),

$$\begin{aligned} \|BT_Y^m(F)\|_{\vartheta,m} &= \sum_{i=1}^m \|T_{\eta_{i1}}\varphi_1 + \cdots + T_{\eta_{im}}\varphi_m\|_{F_{\vartheta}^p} \\ &\leq \sum_{i=1}^m (\|T_{\eta_{i1}}\varphi_1\|_{p,\vartheta} + \cdots + \|T_{\eta_{im}}\varphi_m\|_{p,\vartheta}) \\ &\leq \sum_{i,j=1}^m \|\eta_{ij}\|_{L^\infty(C)} \|\varphi_j\|_{p,\vartheta} \end{aligned}$$

hence, $\|BT_Y^m\|_{\vartheta,m} \leq \|Y\|_{L_m^\infty(C)}$.

The linearity of BT_Y^m follows directly from Equation (3.2). The next lemma characterizes the compactness of BT_Y^m for specific class of matrix-valued function Y .

Lemma 3.4

Assume that $m \in N$ and Y is given by

$$Y = \begin{pmatrix} \eta_{11} & \cdots & \eta_{1m} \\ \vdots & \ddots & \vdots \\ \eta_{m1} & \cdots & \eta_{mm} \end{pmatrix}_{m \times m}$$

where each η_{ij} is defined on C and continuous, and $\eta_{ij}(w) \rightarrow 0$ as $w \rightarrow \infty$. Then BT_Y^m is compact on $F_{\vartheta,m}^p$.

Proof:

Using the continuity of η_{ij} , and since $\eta_{ij}(w) \rightarrow 0$ as $w \rightarrow \infty$, then Theorem 4.2 in [12], shows that each $T_{\eta_{ij}}$ is compact on F_{ϑ}^p . Take a sequence $\{F_\kappa\}_{\kappa=1}^\infty$ which is bounded in $F_{\vartheta,m}^p$, this means that each F_κ is a vector of m functions that contained in F_{ϑ}^p . Now

$$\begin{aligned} BT_Y^m(\{F_\kappa\}_{\kappa=1}^\infty) &= BT_Y^m(F_1), BT_Y^m(F_2), \dots \\ &= \begin{pmatrix} T_{\eta_{11}} & \cdots & T_{\eta_{1m}} \\ \vdots & \ddots & \vdots \\ T_{\eta_{m1}} & \cdots & T_{\eta_{mm}} \end{pmatrix}_{m \times m} \begin{pmatrix} \varphi_{11} \\ \vdots \\ \varphi_{m1} \end{pmatrix}_{m \times 1}, \\ &\quad \begin{pmatrix} T_{\eta_{11}} & \cdots & T_{\eta_{1m}} \\ \vdots & \ddots & \vdots \\ T_{\eta_{m1}} & \cdots & T_{\eta_{mm}} \end{pmatrix}_{m \times m} \begin{pmatrix} \varphi_{12} \\ \vdots \\ \varphi_{m2} \end{pmatrix}_{m \times 1}, \dots \end{aligned}$$

Since $\{F_\kappa\}_{\kappa=1}^\infty$ is bounded, then $\{T_{\eta_{ij}}\varphi_{i\kappa}\}_{\kappa=1}^\infty$ is bounded sequence in F_{ϑ}^p for all $i, j = 1, \dots, m$, but each $T_{\eta_{ij}}$ is compact, thus $\{T_{\eta_{ij}}\varphi_{i\kappa}\}_{\kappa=1}^\infty$ has a convergent subsequence, say $\{T_{\eta_{ij}}\varphi_{i\kappa_k}\}_{\kappa_k=1}^\infty$. This implies that $BT_Y^m(\{F_\kappa\}_{\kappa=1}^\infty)$ has a convergent subsequence in $F_{\vartheta,m}^p$. Hence BT_Y^m is compact on $F_{\vartheta,m}^p$.

3. Fredholm properties of T_Y^m with Y is matrix-valued function

This section specifically investigated the conditions of the block Toeplitz operators that are defined on vector-valued doubling Fock spaces to be Fredholm. The symbol VMO^p is used to represent the space of all locally integrable functions g (that is, functions integrable on every compact subset of C) whose mean oscillation vanishes, where $1 \leq p < \infty$. The symbol VO denotes to the space of all functions that are continuous and of vanishing oscillation. Moreover, the symbol VAP is used to represent the space of all functions that are of vanishing average. A significant property of VMO^p , is that for any function $g \in VMO^p$, $g = g_1 + g_2$, where $g_1 \in VO$ and $g_2 \in VAP$. For more details about these spaces, see [13]. For each $m \in N$, the symbol VO_m denotes the space of all square matrices Y whose entries η_{ij} are essentially bounded functions and $\eta_{ij} \in VO$, for all $1 \leq i, j \leq m$. Similarly, VMO_m^1 denotes the space of all square matrices Y whose entries η_{ij} are essentially bounded functions belonging to VMO^1 , for all $1 \leq i, j \leq m$. In reference [12], Theorem 3.5, Hu and Lv proved that for η is an essential bounded measurable function defined on C , the compactness of the Hankel operator H_η on F_{ϑ}^2 is satisfied if and only if, $\eta \in VMO^1$. Depending on this characterization and Equation (2.1), the following result is obtained.

Lemma 4.1

Assume that $m \in N$ and $Y \in VO_m$ or $Y \in VMO_m^1$. Then the entries of the block Toeplitz operator BT_Y^m are commute modulo compact operators.

Proof.

Since $Y \in VO_m$, then each $\eta_{ij} \in L^\infty(C) \cap VO$. From Theorem 3.5 in [13], each Hankel operator $H_{g_{ij}}$ is compact on F_{ϑ}^2 . From equation (3.1), $T_{\eta_{ij}}T_{\eta_{kn}} - T_{\eta_{ij}\eta_{kn}}$ is compact on F_{ϑ}^2 , for all $\eta_{ij}, \eta_{kn} \in Y$. Hence,

$$T_{\eta_{ij}}T_{\eta_{kn}} - T_{\eta_{kn}}T_{\eta_{ij}} = T_{\eta_{ij}}T_{\eta_{kn}} - T_{\eta_{ij}\eta_{kn}} - (T_{\eta_{kn}}T_{\eta_{ij}} - T_{\eta_{kn}\eta_{ij}})$$

is compact. The next result is obtained by using the previous lemma and is similar to Lemma 21 in [8].

Lemma 4.2

Assume that $m \in N$ and $Y \in VO_m$. Then

$$Det BT_Y^m = T_{Det Y} + K,$$

where, K is compact operator.

Proof.

For any $m \times m$ matrix Y , then the determinant of Y is given by

$$Det(Y) = \sum_{\sigma \in \mu_m} sgn(\delta) \prod_{i=1}^m \eta_{i\delta(i)}$$

where, μ_m the set of m -permutations and $sgn(\delta)$ refers to the sign of each permutation. As BT_Y^m is an $m \times m$ matrix, then

$$Det BT_Y^m = \sum_{\delta \in \mu_m} sgn(\delta) T_{\eta_{1\delta(1)}} T_{\eta_{2\delta(2)}} \cdots T_{\eta_{m\delta(m)}}.$$

Using Lemma 4.2 and for $\eta_i \in F_{\theta}^2$,

$$\begin{aligned} Det(BT_Y^m(F)) &= \sum_{\delta \in \mu_m} sgn(\delta) T_{\eta_{1\delta(1)}} T_{\eta_{2\delta(2)}} \cdots T_{\eta_{m\delta(m)}}(F) \\ &+ K(F) \\ &= P_{\theta} \left(\sum_{\delta \in \mu_m} sgn(\delta) \eta_{1\delta(1)} \eta_{2\delta(2)} \cdots \eta_{m\delta(m)}(F) \right) + K(F) \\ &= T_{Det(Y)}(F) + K(F), \text{ for all } F \in F_{\theta, m}^p \end{aligned}$$

In [12], Theorem 2.1, the authors proved that η is an essential bounded measurable function on C and $\eta \in VO$, then T_{η} is Fredholm on F_{θ}^2 if and only if for some $\alpha > 0$, $|\eta(\omega)| > \alpha$, for all ω outside the disk $D(0, \alpha)$. Notably, for any block operator $BT^m = (T_{ij})_{i,j=1}^m$ defined on the Banach space $\mathcal{E}_m$, it is known that if all entries of BT^m that are commute modulo compact operators, then BT^m is Fredholm if and only if $Det(BT^m)$, Theorem 2.1, [14]. Using these two significant results, we proved our main theorem in this section.

Theorem 4.3

Hypothesize that $m \in N$ and Y is a matrix-valued symbol such that $Y \in VO_m$. Then the necessary and

the sufficient condition for the Fredholmness of BT_Y^m on $F_{\theta, m}^2$ is given by

$$|Det(Y)(\omega)| > \alpha > 0, \text{ for all } \omega \text{ outside the disk } D(0, \alpha).$$

Proof.

Since $Y \in VO_m$, then each $\eta_{ij} \in L^{\infty}(C) \cap VO$. Using Theorem 2.1 in [12], $T_{\eta_{ij}}$ is Fredholm on F_{θ}^2 if and only if $|\eta_{ij}(\omega)| > \alpha$, for all ω outside the disk $D(0, \alpha)$, for all $1 \leq i, j \leq m$. Thus $T_{Det(Y)}$ is Fredholm if and only if $|Det(Y)(\omega)| > \alpha > 0$, for all ω outside the disk $D(0, \alpha)$. From the previous lemma, $Det(BT_Y^m)$ is Fredholm if and only if $T_{Det(Y)}$ is Fredholm, hence $Det(BT_Y^m)$ is Fredholm if and only if $|Det(Y)(\omega)| > \alpha > 0$, for all ω outside the disk $D(0, \alpha)$. Lemma 4.2 shows that the elements of BT_Y^m are commute modulo compact operators. Therefore, by using Theorem 2.1 in [14], BT_Y^m is Fredholm if and only if $Det(BT_Y^m)$ is Fredholm which means that BT_Y^m is Fredholm if and only if $|Det(Y)(\omega)| > \alpha > 0$, for all ω outside the disk $D(0, \alpha)$. In [15], Theorem 4.9, Hu and Virtanen gave the necessary and the sufficient condition for the operator T_{η} to be Fredholm on F_{θ}^2 , where $\eta \in VMO^1$. This condition is

$$0 < \liminf_{|\omega| \rightarrow \infty} |\tilde{\eta}(\omega)| \leq \limsup_{|\omega| \rightarrow \infty} |\tilde{\eta}(\omega)| < \infty \quad \dots (4.1)$$

where, $\tilde{\eta}$ is the Berezin transform of η , see [14].

Using this outcome, the following theorem is achieved.

Theorem 4.4

Hypothesize that $m \in N$ and Y is matrix-valued symbol such that $Y \in VMO_m^1$. Then the necessary and the sufficient condition for the Fredholmness of BT_Y^m on $F_{\theta, m}^2$ is given by

$$\begin{aligned} 0 &< \liminf_{|\omega| \rightarrow \infty} |\widetilde{Det(Y)}(\omega)| \\ &\leq \limsup_{|\omega| \rightarrow \infty} |\widetilde{Det(Y)}(\omega)| \\ &< \infty \quad \dots (4.2) \end{aligned}$$

Proof.

Since $Y \in VMO_m^1$, then each $g_{ij} \in L^{\infty}(C) \cap VMO^1$. Using Theorem 4.9 in [15], $T_{\eta_{ij}}$ is Fredholm on F_{θ}^2 if and only if (4.1) holds for η_{ij} , $1 \leq i, j \leq m$. Thus $T_{Det Y}$ is Fredholm if and only if $Det(Y)$ satisfies (4.2). From the lemma 3.3, $Det(BT_Y^m)$ is Fredholm if and only if $T_{Det Y}$ is Fredholm, hence $Det(BT_Y^m)$ is Fredholm if and only if $Det(Y)$ satisfies (4.2). Lemma 4.2 shows that the elements of BT_Y^m are commute modulo

compact operators. Therefore, by using Theorem 2.1 in [14], BT_Y^m is Fredholm if and only if $\text{Det}(BT_Y^m)$ is Fredholm which means that BT_Y^m is Fredholm if and only if $\text{Det}(Y)$ satisfies (4.2).

5. Conclusions and Recommendations

For a subharmonic function $0 \neq \vartheta$ on C and $1 \leq m < \infty$, the vector-valued doubling Fock spaces $F_{\vartheta,m}^p$ are defined to be the collection of all vector-valued functions $F_{m \times 1}$, where the component functions are in F_{ϑ}^p . This article introduces the block Toeplitz operator BT_Y^m on $F_{\vartheta,m}^p$, where the symbol $Y = (\eta_{ij})_{1 \leq i,j \leq m}$ is a matrix-valued function and each η_{ij} is a function in $L^\infty(C)$. The investigation showed that BT_Y^m is bounded on $F_{\vartheta,m}^p$ with regard to this symbol Y . Moreover, with some conditions on Y , the compactness and Fredholm properties of BT_Y^m on $F_{\vartheta,m}^p$ are thus characterized. A vital extension to this work is the calculation of the index formula of BT_Y^m when it is Fredholm and this requires more conditions on the symbol Y that make all commutators of the interties of BT_Y^m is a finite rank or trace class, [3]. Furthermore, studying the normality and hyponormality of BT_Y^m on vector-valued Fock space $F_{\vartheta,m}^p$ will be Fascinating. However, another interesting work is to find a solution for the operator equation $AT + T^*B = C$, where A, B are two Toeplitz operators on Fock spaces. The solution of this operator equation has been investigated in [16], where A, B are two bounded linear normal operators on Hilbert space.

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