The Effect of Disease and Harvest in the Eco-Epidemical Food Chain Model

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Abstract

This paper describes a three species food web model with linear functional response and incidence disease. This model consisting of a prey, intermediate predator and top predator where harvesting of top predator species and SIS disease spread in prey are taken into consideration. The stability analysis of all possible equilibrium points are carried out. We discussed the effect both of harvest and disease on the stability of this model. Finally, we used the numerical simulations to verify the analytical results.

Keywords: SIS epidemics disease, prey-Predator model, Chain of food, Harvest management; stability.

Introduction

Mathematics is one way to explain many of the ideas and concepts in the sciences. In the field of food webs play a very important role in ecology, a lot of theoretical studies were carried out since the beginning of last century to explain the interaction between the ecological communities. One particular study describes the interaction between one population (prey) and the other (predator) living in a closed environment with the three populations striving for survival, for example see [1-6]. On the other hand densely populated areas are a good incubator for the spread of infectious diseases. Therefore, there is increasing opportunity for the spread of diseases among the communities interacting with each other [7-18]. There are numerous studies on the effects of harvesting on population growth. In the context of predator-prey interaction, some studies that treat the populations being harvested as a homogeneous resource include those [19-26]. In this paper, we proposed and analyzed a three species food web model in which the prey follows the susceptible-infected-susceptible cycle and the top predator is harvested. In this model, we used linear form as a functional response and disease incidence for describing the transition of diseases. To be followed by a study on the stability of the equilibrium points. Next, we discuss the nature of the solutions and finally the numerical simulations to support the model.

Mathematical Model

We consider the following system as a model simulating a tritophic level food chain. The dynamics of three species food chain model with linear type of function response is governed by the following differential equation, where $N(T)$ is the population density of the lowest trophic level species (prey) at time T, there is SIS (Susceptible-Infected-Susceptible) epidemic disease spread among the prey population and it transmitted between the prey individuals by contact, according to linear incidence rate with infection rate constant $h_3 > 0$. The infected prey can be recovered and become susceptible again with recovery rate constant $h_5 > 0$. Therefore, the total prey population is divided into two classes: susceptible individuals that is denoted by $S(T)$, and infected individuals that is denoted by $I(T)$. Hence at any time T the total prey population is $N(T) = S(T) + I(T)$. $X(T)$ is the population density of the middle trophic level species (intermediate predator) at time T and $Y(T)$ is the population density of highest trophic level species (top predator) at time T.

$$\begin{aligned}
\frac{dS}{dT} &= S\left[h_1 - h_2(S+I) - h_3 I - h_4 X\right] + h_5 I \\
\frac{dI}{dT} &= I\left(h_3 S - h_6 X - h_5 - h_7\right) \\
\frac{dX}{dT} &= X\left(h_8 - h_9 X + h_{10} S + h_{11} I - h_{12} Y\right) \\
\frac{dY}{dT} &= Y\left[h_{13} - h_{14} Y + h_{15} X - qE\right]
\end{aligned} \qquad .... (1)$$

It is assumed that, all the model parameters are positive values. The prey $N(T)$ grows with intrinsic growth rate h_1 and carrying capacity $h_1h_2^{-1}$ in absence of predation. The intermediate predator grows logistically with intrinsic growth rate h_8 and carrying capacity $h_8h_9^{-1}$, also the top predator grows logistically with intrinsic growth rate h_{13} and carrying capacity $h_{13}h_{14}^{-1}$. The predator X consumes the prey S and I with maximum attack rates h_4 and h_6 respectively, while predator Y preys upon X according to maximum attack rate h_{12}. The parameters h_{10}, h_{11} and h_{15} are conversion rates of prey to predator for species X and Y respectively, for which $h_4 > h_{10}$, $h_6 > h_{11}$ and $h_{12} > h_{15}$. The parameter h_7 is disease induced mortality rate for species I. Finally, $q > 0$ is the catch ability co-efficient of the predator, $E > 0$ is the harvesting effort and qEY is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis.

The Jacobian matrix of system (1) is $J = (\beta_{ij}) \in \Re^{4\times4}$, with entries

$$\beta_{11} = h_1 - 2h_2S - h_2I - h_3I - h_4X\,;\ \beta_{13} = -h_4S\,;$$
$$\beta_{12} = h_5 - h_2S - h_3S\,;\ \beta_{14} = 0\,;\ \beta_{21} = h_3I\,;$$
$$\beta_{22} = h_3S - h_6X - h_5 - h_7;\ \beta_{23} = -h_6I\,;$$
$$\beta_{24} = 0\,;\ \beta_{31} = h_{10}X\,;\ \beta_{32} = h_{11}X\,;\ \beta_{34} = -h_{12}X\,;$$
$$\beta_{33} = h_8 - 2h_9X + h_{10}S + h_{11}I - h_{12}Y\,;$$
$$\beta_{41} = \beta_{42} = 0\,;\ \beta_{43} = h_{15}Y\,;$$
$$\beta_{44} = h_{13} - 2h_{14}Y + h_{15}X - qE$$

By introducing the total environment population $P(T) = S(T) + I(T) + X(T) + Y(T)$, summing the equation (1) and bounding the right-hand from above, following the steps of [27], boundedness of the solution trajectories of this model is established. In particular,

$$\lim_{T\to\infty}(S(T) + I(T)) \le \frac{h_1}{h_2}\,;\quad \lim_{T\to\infty} X(T) \le \frac{h_8}{h_9}\,;$$
$$\lim_{T\to\infty} Y(T) \le \frac{h_{13}}{h_{14}} \text{ and } \lim_{T\to\infty} P(T) \le \varphi\left(\frac{h_1h_9 + h_2h_8}{h_2h_9}\right)$$

where $0 < \varphi < \min\{h_7\ , qE\}$

In what follows, the system's equilibria are E_k and we denote by J_k and $\beta_{ij}^{[k]}$ the Jacobian and its entries evaluated at E_k, $i = 1,2,3,4$, $j = 1,2,3,4,\ k = 0,1,\dots,11$.

Analysis of System

Clearly the origin equilibrium point $E_0 = (0,0,0,0)$ is a trivial solution of the system (1), and the Jacobian matrix at E_0 becomes a triangular matrix, then the eigenvalues are $\lambda_S^{[0]} = h_1 > 0$, $\lambda_X^{[0]} = h_8 > 0$ $\lambda_I^{[0]} = -(h_5 + h_7) < 0$ and $\lambda_Y^{[0]} = h_{13} - qE$. Yield, E_0 is a saddle point (unstable). By the same way, the Jacobian matrix at the equilibrium point $E_1 = (S_1,0,0,0)$ where $S_1 = h_1h_2^{-1}$ becomes a triangular matrix, and the eigenvalues are

$$\lambda_S^{[1]} = -h_1 < 0, \lambda_I^{[1]} = h_3S_1 - (h_5 + h_7),$$
$$\lambda_X^{[1]} = h_8 + h_{10}S_1 > 0 \text{ and } \lambda_Y^{[1]} = h_{13} - qE\,.$$

Therefore the equilibrium point E_1 is always saddle point (unstable). The equilibrium point $E_2 = (0,0,X_2,0)$ where $x_2 = h_8h_9^{-1}$ has the following characteristic equation:

$$\begin{aligned} P(\lambda^{[2]}) = &\left((h_1 - h_4X_2) - \lambda_S^{[2]}\right)\left(-h_8 - \lambda_X^{[2]}\right) \\ &\times\left((h_{13} + h_{15}X_2 - qE) - \lambda_Y^{[2]}\right) \\ &\times\left(-(h_6X_2 + h_5 + h_7) - \lambda_I^{[2]}\right) = 0 \end{aligned}$$

So, the eigenvalues are $\lambda_I^{[2]} = -(h_6X_2 + h_5 + h_7) < 0$

$$\lambda_S^{[2]} = h_1 - h_4X_2, \lambda_X^{[2]} = -h_8 < 0 \text{ and}$$
$$\lambda_Y^{[2]} = h_{13} + h_{15}X_2 - qE\,.$$

Therefore the equilibrium point E_2 is locally asymptotically stable if and only if $h_1h_4^{-1} < X_2 < (qE - h_{13})h_{15}^{-1}$ and $h_{13} < qE$ hold. Otherwise, E_2 is saddle point (unstable). The equilibrium point is $E_3 = (0,0,0,Y_3)$ where $Y_3 = (h_{13} - qE)h_{14}^{-1}$ and the necessary and sufficient condition for existence E_3 is $h_{13} > qE$, the eigenvalues are

$$\lambda_X^{[3]} = h_8 - h_{12}Y_3$$
$$\lambda_S^{[3]} = h_1 > 0, \lambda_I^{[3]} = -(h_5 + h_7) < 0, \text{ and}$$
$$\lambda_Y^{[3]} = -(h_{13} - qE) < 0\,.$$

Therefore the equilibrium point E_3 is saddle point (unstable). The equilibrium point $E_4 = (S_4, I_4, 0, 0)$ where $S_4 = (h_5 + h_7)h_3^{-1}$ and $I_4 = \frac{S_4(h_1 - h_2 S_4)}{(h_2 S_4 + h_7)}$, the necessary and sufficient condition for existence E_4 is $h_1 h_2^{-1} > S_4$, and the characteristic equation is:

$$P(\lambda^{[4]}) = \left((h_8 + h_{10}S_4 + h_{11}I_4) - \lambda_X^{[4]}\right)\left((h_{13} - qE) - \lambda_Y^{[4]}\right) \times \begin{pmatrix} \lambda^2 - (h_1 - 2h_2 S_4 - (h_2 + h_3)I_4)\lambda \\ + h_3 I_4 (h_2 S_4 + h_7) \end{pmatrix} = 0$$

So, the eigenvalues are $\lambda_X^{[4]} = h_8 + h_{10}S_4 + h_{11}I_4 > 0$, $\lambda_Y^{[4]} = h_{13} - qE$ and $\lambda_S^{[4]} \times \lambda_I^{[4]} = h_3 I_4 (h_2 S_4 + h_7) > 0$
$\lambda_S^{[4]} + \lambda_I^{[4]} = h_1 - 2h_2 S_4 - h_2 I_4 - h_3 I_4$.

Therefore the equilibrium point E_4 is saddle point (unstable). Now, we turn to the investigation of equilibria $E_k = (S_k, I_k, X_k, Y_k), k = 5,6,\ldots,11$ of system (1). The equilibrium points in which simple prey-predator model namely $E_5 = (S_5, 0, X_5, 0)$ and $E_7 = (0, 0, X_7, Y_7)$ where

$S_5 = \frac{(h_1 h_9 - h_4 h_8)}{(h_4 h_{10} + h_2 h_9)}$, $X_5 = \frac{(h_1 h_{10} + h_2 h_8)}{(h_4 h_{10} + h_2 h_9)}$ with the feasibility conditions $h_4 h_8 < h_1 h_9$, and $X_7 = \frac{(h_8 h_{14} + h_{12} qE - h_{12} h_{13})}{(h_9 h_{14} - h_{12} h_{15})}$,
$Y_7 = \frac{(h_8 - h_9 X_7)}{h_{12}}$ with feasibility conditions $h_8 h_{14} + h_{12} qE > h_{12} h_{13}$ and $X_7 < h_8 h_9^{-1}$

respectively. The equilibrium point $E_6 = (S_6, 0, 0, Y_6)$ where $S_6 = h_1 h_2^{-1}$ and $Y_6 = (h_{13} - qE)h_{14}^{-1}$ with the feasibility condition $h_{13} > qE$. Also, we have Eco-Epidemiological model with equilibrium point $E_8 = (S_8, I_8, X_8, 0)$ where

$$X_8 = \frac{(h_3 S_8 - h_5 - h_7)}{h_6},$$
$$I_8 = \frac{(h_3 h_9 - h_6 h_{10})S_8 - (h_6 h_8 + h_9 (h_5 + h_7))}{h_6 h_{11}}$$

while S_8 represents a positive root of the equation $A_1 S^2 + A_2 S + A_3 = 0$ where

$$A_1 = -\begin{bmatrix} h_2 h_6 h_{11} + (h_2 + h_3)(h_3 h_9 - h_6 h_{10}) \\ + h_3 h_4 h_{11} \end{bmatrix} < 0$$
$$A_2 = h_1 h_6 h_{11} + (h_2 + h_3)[h_6 h_8 + h_9 (h_5 + h_7)] + h_4 h_{11}(h_5 + h_7) + h_5 (h_3 h_9 - h_6 h_{10}) > 0$$
$$A_3 = -h_5 [h_6 h_8 + h_9 (h_5 + h_7)] < 0$$

Obviously, E_8 exists uniquely in the interior of the first octant of $SIX-$space if and only if $h_3 h_9 > h_6 h_{10}$ and $S_8 > \max\left\{\frac{(h_5 + h_7)}{h_3}, \frac{h_6 h_8 + h_9 (h_5 + h_7)}{(h_3 h_9 - h_6 h_{10})}\right\}$.

The equilibrium point $E_9 = (S_9, I_9, 0, Y_9)$ where components

$S_9 = \frac{(h_5 + h_7)}{h_3}$, $I_9 = \frac{(h_1 - h_2 S_9)S_9}{(h_7 + h_2 S_9)}$ and $Y_9 = \frac{(h_{13} - qE)}{h_{14}}$ with feasibility conditions $S_9 < \frac{h_1}{h_2}$ and $h_{13} > qE$. We have food chain model with equilibrium point

$E_{10} = (S_{10}, 0, X_{10}, Y_{10})$ where:

$$S_{10} = \frac{\begin{bmatrix} h_1 h_9 h_{14} + h_4 h_{12} h_{13} + h_1 h_{12} h_{15} \\ - h_4 h_8 h_{14} - h_4 h_{12} qE \end{bmatrix}}{[h_2 h_9 h_{14} + h_4 h_{10} h_{14} + h_2 h_{12} h_{15}]},$$
$$X_{10} = \frac{(h_1 - h_2 S_{10})}{h_4}, \quad Y_{10} = \frac{(h_{13} + h_{15} X_{10} - qE)}{h_{14}}$$

with feasibility conditions
$h_4 h_8 h_{14} + h_4 h_{12} qE < h_1 h_9 h_{14} + h_4 h_{12} h_{13} + h_1 h_{12} h_{15}$,
$S_{10} < h_1 h_2^{-1}$
and $h_{13} + h_{15} X_{10} > qE$. Finally, we have the coexistence equilibrium point $E_{11} = (S_{11}, I_{11}, X_{11}, Y_{11})$ with components

$$S_{11} = \frac{(h_6 X_{11} + h_5 + h_7)}{h_3}, \quad Y_{11} = \frac{(h_{15} X_{11} + h_{13} - qE)}{h_{14}}$$
$$I_{11} = \frac{S_{11}(h_1 - h_2 S_{11} - h_4 X_{11})}{(h_2 S_{11} + h_6 X_{11} + h_7)},$$

while X_{11} represents a positive root of the following second order polynomial equation $A_1 X^2 + A_2 X + A_3 = 0$ where

$$A_1 = h_6\left[h_6h_{10}h_{14} - h_3h_9h_{14} - h_{12}h_{15}\right](h_2 - h_3) + h_6h_{11}h_{14}(h_2h_6 - h_3h_4)$$
$$A_2 = (h_2h_5 + h_2h_7 + h_3h_7)\left[h_6h_{10}h_{14} - h_3h_9h_{14} - h_{12}h_{15}\right] + h_6h_{11}h_{14}\left[h_1h_3 - 2h_2(h_5 + h_7)\right] + \left[h_5h_6h_{10}h_{14} + h_6h_7h_{10}h_{14} - h_6h_{12}h_{13} + h_6h_{12}qE\right] \times (h_2 - h_3) - h_3h_4h_{11}h_{14}(h_5 + h_7)$$
$$A_3 = \left[h_3h_8h_{14} + h_5h_{10}h_{14} + h_7h_{10}h_{14} - h_{12}h_{13} + h_{12}qE\right] \times (h_2h_5 + h_2h_7 + h_3h_7) + h_3h_6h_8h_{14}(h_2 - h_3) + (h_5 + h_7)\left[h_1h_3 - h_2(h_5 + h_7)\right]$$

Therefore, straight forward computation shows that E_{11} exists uniquely in the int. $\Re^4_+$ if and only if $h_{13} + h_{15}X_{11} > qE$, $h_1 > h_2S_{11} + h_4X_{11}$ and one set the following conditions holds

$$A_1 > 0 \quad and \quad A_3 < 0$$
$$or$$
$$A_1 < 0 \quad and \quad A_3 > 0$$

Theorem (1):

The equilibrium point E_5 is locally asymptotically stable in $\Re^4_+$ if and only if:

$$h_{13} + h_{15}X_5 < qE,\ h_3S_5 < h_6X_5 + h_5 + h_7$$
$$2h_8 < \min\left\{h_2S_5 + 3h_9X_5\ ,\ \frac{\left[h_4h_{10} + 3h_2h_9\right]x_5}{h_2}\right\} \ldots.. (2)$$

***Proof*:**

The Jacobian matrix of the system (1) at E_5 is given by:

$$J_5 = \begin{pmatrix} -h_2S_5 & \begin{pmatrix} h_5 - h_2S_5 \\ -h_3S_5 \end{pmatrix} & -h_4S_5 & 0 \\ 0 & \begin{pmatrix} h_3S_5 - h_6X_5 \\ -h_5 - h_7 \end{pmatrix} & 0 & 0 \\ h_{10}X_5 & h_{11}X_5 & (2h_8 - 3h_9X_5) & -h_{12}X_5 \\ 0 & 0 & 0 & \begin{pmatrix} h_{15}X_5 \\ h_{13} - qE \end{pmatrix} \end{pmatrix}$$

So, the characteristic equation of J_5 can be written by

$$P\left(\lambda^{[5]}\right) = \begin{pmatrix} \left(\lambda^{[5]}\right)^2 + (h_2S_5 - 2h_8 + 3h_9X_5)\lambda^{[5]} \\ + S_5\left(h_4h_{10}X_5 - h_2(2h_8 - 3h_9X_5)\right) \end{pmatrix} \times \left((h_3S_5 - h_6X_5 - h_5 - h_7) - \lambda_I^{[5]}\right) \times \left((h_{13} + h_{15}X_5 - qE) - \lambda_Y^{[5]}\right)$$

from which, we obtain that:

$$\lambda_I^{[5]} = h_3S_5 - h_6X_5 - h_5 - h_7,\ \lambda_Y^{[5]} = h_{13} + h_{15}X_5 - qE$$
$$\lambda_S^{[5]} + \lambda_X^{[5]} = -(h_2S_5 - 2h_8 + 3h_9X_5),$$
$$\lambda_S^{[5]} \times \lambda_X^{[5]} = S_5\left[h_4h_{10}X_5 - h_2(2h_8 - 3h_9x_5)\right]$$

Here $\lambda_S^{[5]}, \lambda_I^{[5]}, \lambda_X^{[5]}$ and $\lambda_Y^{[5]}$ denote to the eigenvalues in the $S-$direction, $I-$direction, $X-$direction and $Y-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (2) holds. Therefore, the equilibrium point E_5 is locally asymptotically stable in $\Re^4_+$.

Theorem (2):

The equilibrium point E_6 is locally asymptotically stable in $\Re^4_+$ if and only if:

$$\begin{aligned} & h_8 + h_{10}S_6 < h_{12}Y_6 \\ & h_3S_6 < (h_5 + h_7) \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

***Proof*:**

The Jacobian matrix of the system (1) at E_6 is given by:

$$J_6 = \begin{pmatrix} -h_1 & \begin{pmatrix} h_5 - h_2S_6 \\ -h_3S_6 \end{pmatrix} & -h_4S_6 & 0 \\ 0 & \begin{pmatrix} h_3S_6 \\ -h_5 - h_7 \end{pmatrix} & 0 & 0 \\ 0 & 0 & \begin{pmatrix} h_8 + h_{10}S_6 \\ -h_{12}Y_6 \end{pmatrix} & 0 \\ 0 & 0 & h_{15}Y_6 & -(h_{13} - qE) \end{pmatrix}$$

So, the characteristic equation of J_6 can be written by

$$P\left(\lambda^{[6]}\right) = \left((h_3S_6 - h_5 - h_7) - \lambda_I^{[6]}\right)\left(-(h_{13} - qE) - \lambda_Y^{[6]}\right) \times \left((h_8 + h_{10}S_6 - h_{12}Y_6) - \lambda_X^{[6]}\right)\left(-h_1 - \lambda_S^{[6]}\right)$$

from which, we obtain that:

$$\lambda_S^{[6]} = -h_1 < 0\ ,\ \lambda_I^{[6]} = (h_3S_6 - h_5 - h_7),$$
$$\lambda_X^{[6]} = (h_8 + h_{10}S_6 - h_{12}Y_6), \lambda_Y^{[5]} = -(h_{13} - qE) < 0$$

Here $\lambda_S^{[6]}, \lambda_I^{[6]}, \lambda_X^{[6]}$ and $\lambda_Y^{[6]}$ denote to the eigenvalues in the $S-$direction, $I-$direction, $X-$direction and $Y-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (3) holds. Therefore, the equilibrium point E_6 is locally asymptotically stable in $\Re^4_+$.

Theorem (3):

The equilibrium point E_7 is locally asymptotically stable in $\Re^4_+$ if and only if $h_1h_4^{-1} < X_7$.

***Proof*:**

The Jacobian matrix of the system (1) at E_7 is given by:

$$J_7=\begin{pmatrix} \begin{pmatrix} h_1 \\ -h_4X_7 \end{pmatrix} & h_5 & 0 & 0 \\ 0 & -\begin{pmatrix} h_6X_7 \\ +h_5+h_7 \end{pmatrix} & 0 & 0 \\ h_{10}X_7 & h_{11}X_7 & -h_9X_7 & -h_{12}X_7 \\ 0 & 0 & h_{15}Y_7 & -h_{14}Y_7 \end{pmatrix}$$

So, the characteristic equation of J_7 can be written by

$$P(\lambda^{[7]})=\left((h_1-h_4X_7)-\lambda_S^{[7]}\right)\left(-(h_6X_7+h_5+h_7)-\lambda_I^{[7]}\right) \times\begin{pmatrix} (\lambda^{[7]})^2+(h_9X_7+h_{14}Y_7)\lambda^{[7]} \\ +(h_9h_{14}+h_{12}h_{15})X_7Y_7 \end{pmatrix}$$

from which, we obtain that

$$\lambda_S^{[7]}=h_1-h_4X_7\ ,\ \lambda_I^{[7]}=-(h_6X_7+h_5+h_7)<0$$
$$\lambda_X^{[7]}+\lambda_Y^{[7]}=-(h_9X_7+h_{14}Y_7)<0\ ,$$
$$\lambda_X^{[7]}\times\lambda_Y^{[7]}=h_9h_{14}+h_{12}h_{15}>0$$

Here $\lambda_S^{[7]},\lambda_I^{[7]},\lambda_X^{[7]}$ and $\lambda_Y^{[7]}$ denote to the eigenvalues in the $S-$direction, $I-$direction, $X-$direction and $Y-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if $h_1h_4^{-1}<X_7$ holds. Therefore, the equilibrium point E_7 is locally asymptotically stable in $\Re_+^4$.

Theorem (4):

If the following conditions hold

$$h_{13}+h_{15}X_8<qE \quad\text{.......................................}(4a)$$

$$\left.\begin{aligned} & h_1<2h_2S_8+h_3I_8+h_4X_8 \\ & h_3h_4S_8<h_6h_9X_8 \end{aligned}\right\} \quad\text{........................}(4b)$$

Then, the equilibrium point E_8 is a locally asymptotically stable.

***Proof*:**

The Jacobian matrix of the system (1) at E_8 is given by:

$$J_8=\begin{pmatrix} \begin{pmatrix} h_1-2h_2S_8 \\ -h_2I_8-h_3I_8 \\ -h_4X_8 \end{pmatrix} & \begin{pmatrix} h_5-h_2S_8 \\ -h_3S_8 \end{pmatrix} & -h_4S_8 & 0 \\ h_3I_8 & 0 & -h_6I_8 & 0 \\ h_{10}X_8 & h_{11}X_8 & -h_9X_8 & -h_{12}X_8 \\ 0 & 0 & 0 & \begin{pmatrix} h_{13}+ \\ h_{15}X_8-qE \end{pmatrix} \end{pmatrix}$$

So, the characteristic equation of J_8 can be written by

$$P(\lambda^{[8]})=\left(\lambda_Y^{[8]}-(h_{13}+h_{15}X_8-qE)\right) \times\left[(\lambda^{[8]})^3+F_1(\lambda^{[8]})^2+F_2(\lambda^{[8]})+F_3\right] \text{ with}$$

$$F_1=-h_1+2h_2S_8+(h_2+h_3)I_8+(h_4+h_9)X_8$$
$$\begin{aligned} F_2&=h_3I_8(h_2S_8+h_3S_8-h_5)+h_2h_9I_8 \\ &\quad -h_9X_8(h_1-2h_2S_8-h_3I_8-h_4X_8) \\ &\quad +h_4h_{10}S_8X_8+h_6h_{11}I_8X_8 \end{aligned}$$

$$F_3=I_8X_8\begin{bmatrix} (h_3h_9-h_6h_{10})(h_2S_8+h_3S_8-h_5) \\ -h_6h_{11}(h_1-2h_2S_8-(h_2+h_3)I_8-h_4X_8) \\ +h_3h_4h_{11}S_8 \end{bmatrix}$$

$$\begin{aligned} \Delta=F_1F_2-F_3&=X_8\begin{bmatrix} h_1-2h_2S_8-(h_2+h_3)I_8 \\ -(h_4+h_9)X_8 \end{bmatrix} \\ &\quad\times\left[h_9(h_1-2h_2S_8-(h_2+h_3)I_8-h_4X_8)-h_4h_{10}S_8\right] \\ &\quad+\left[h_3(h_1-2h_2S_8-(h_2+h_3)I_8-h_4X_8)-h_6h_{10}X_8\right] \\ &\quad\times I_8(h_5-h_2S_8-h_3S_8)+h_{11}X_8I_8(h_6h_9X_8-h_3h_4S_8) \end{aligned}$$

Here $\lambda_Y^{[8]}$ denote to the eigenvalue in the $Y-$direction. The Routh-Hurwitz conditions require $F_i>0\ \forall i=1,3$ and $\Delta=F_1F_2-F_3>0$, follows from condition (4b) and in addition the negativity of the other eigenvalues, namely condition (4a). So, according to Routh-Hurwitz criterion E_8 is locally asymptotically stable.

Theorem (5):

The equilibrium point E_9 is locally asymptotically stable in $\Re_+^4$ if and only if

$$\begin{aligned} & (h_8+h_{10}S_9+h_{11}I_9)h_{12}^{-1}<Y_9 \\ & h_1<2h_2S_9+(h_2+h_3)I_9 \end{aligned} \quad\text{........................}(5)$$

***Proof*:**

The Jacobian matrix of the system (1) at E_9 is given by:

$$J_9=\begin{pmatrix} \beta_{11}^{[9]} & \beta_{12}^{[9]} & \beta_{13}^{[9]} & 0 \\ \beta_{21}^{[9]} & 0 & \beta_{23}^{[9]} & 0 \\ 0 & 0 & \beta_{33}^{[9]} & 0 \\ 0 & 0 & \beta_{43}^{[9]} & \beta_{44}^{[9]} \end{pmatrix}$$

where:

$$\beta_{11}^{[9]}=h_1-2h_2S_9-(h_2+h_3)I_9\ ;\ \beta_{12}^{[9]}=-(h_2S_9+h_7)$$
$$\beta_{13}^{[9]}=-h_4S_9\ ;\ \beta_{21}^{[9]}=h_3I_9\ ;\ \beta_{23}^{[9]}=-h_6I_9\ ;$$
$$\beta_{33}^{[9]}=h_8+h_{10}S_9+h_{11}I_9-h_{12}Y_9\ ;$$
$$\beta_{43}^{[9]}=h_{15}Y_9\ ;\ \beta_{44}^{[9]}=-(h_{13}-qE)$$

So, the characteristic equation of J_9 can be written by

$$P\left(\lambda^{[9]}\right)=\left(\left(h_8+h_{10}S_9+h_{11}I_9-h_{12}Y_9\right)-\lambda_X^{[9]}\right) \times\begin{pmatrix}\left(\lambda^{[9]}\right)^2-\left(h_1-2h_2S_9-\left(h_2+h_3\right)I_9\right)\lambda^{[9]} \\ +h_3I_9\left(h_2S_9+h_7\right)\end{pmatrix} \times\left(-\left(h_{13}-qE\right)-\lambda_Y^{[9]}\right)$$

from which, we obtain that:

$$\lambda_X^{[9]}=h_8+h_{10}S_9+h_{11}I_9-h_{12}Y_9\ ,$$
$$\lambda_Y^{[9]}=-\left(h_{13}-qE\right)<0$$

$$\lambda_S^{[9]}+\lambda_I^{[9]}=h_1-2h_2S_9-\left(h_2+h_3\right)I_9\ ,$$
$$\lambda_S^{[9]}\times\lambda_I^{[9]}=h_3I_9\left(h_2S_9+h_7\right)>0$$

Here $\lambda_S^{[9]},\lambda_I^{[9]},\lambda_X^{[9]}$ and $\lambda_Y^{[9]}$ denote to the eigenvalues in the $S-$direction, $I-$direction, $X-$direction and $Y-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (5) holds. Therefore, the equilibrium point E_9 is locally asymptotically stable in $\Re_+^4$.

<u>Theorem (6):</u>

The equilibrium point E_{10} is locally asymptotically stable in $\Re_+^4$ if and only if

$$S_{10}<\left(h_5+h_7+h_6X_{10}\right)h_3^{-1} \quad \text{.........................(6a)}$$
$$h_8+h_{10}S_{10}<2h_9X_{10}+h_{12}Y_{10} \quad \text{...................(6b)}$$

***<u>Proof</u>*:**

The Jacobian matrix of the system (1) at E_{10} is given by:

$$J_{10}=\begin{pmatrix}\beta_{11}^{[10]} & \beta_{12}^{[10]} & \beta_{13}^{[10]} & 0 \\ 0 & \beta_{22}^{[10]} & 0 & 0 \\ \beta_{31}^{[10]} & \beta_{32}^{[10]} & \beta_{33}^{[10]} & \beta_{34}^{[10]} \\ 0 & 0 & \beta_{43}^{[10]} & \beta_{44}^{[10]}\end{pmatrix}$$

where:

$\beta_{11}^{[10]}=-h_2S_{10};\beta_{12}^{[10]}=h_5-h_2S_{10}-h_3S_{10};$
$\beta_{13}^{[10]}=-h_4S_{10}\ ;\ \beta_{31}^{[10]}=h_{10}X_{10}\ ;\ \beta_{32}^{[10]}=h_{11}X_{10};$
$\beta_{22}^{[10]}=h_3S_{10}-h_6X_{10}-h_5-h_7\ ;\ \beta_{34}^{[10]}=-h_{12}X_{10};$
$\beta_{33}^{[10]}=h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10};$
$\beta_{43}^{[10]}=h_{15}Y_{10}\ ;\ \beta_{44}^{[10]}=-\left(h_{13}+h_{15}X_{10}-qE\right)$

So, the characteristic equation of J_{10} can be written by

$$P\left(\lambda^{[10]}\right)=\left(\lambda_I^{[10]}-\left(h_3S_{10}-h_6X_{10}-h_5-h_7\right)\right) \times\left[\left(\lambda^{[10]}\right)^3+F_1\left(\lambda^{[10]}\right)^2+F_2\left(\lambda^{[10]}\right)+F_3\right]$$

with

$$F_1=-\left(h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10}\right)+h_2S_{10}+h_{14}Y_{10}$$
$$F_2=h_2h_{14}S_{10}Y_{10}+X_{10}\left(h_{12}h_{15}Y_{10}+h_4h_{10}S_{10}\right)-\left(h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10}\right)\times\left(h_1S_{10}+h_{14}Y_{10}\right)$$
$$F_3=S_{10}Y_{10}\begin{bmatrix}\left(h_2h_{12}h_{15}+h_4h_{10}h_{14}\right)X_{10}-h_2h_{14} \\ \times\left(h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10}\right)\end{bmatrix}$$

$$\Delta=F_1F_2-F_3$$
$$=h_4h_{10}S_{10}X_{10}\left[h_2S_{10}-\begin{pmatrix}h_8-2h_9X_{10} \\ +h_{10}S_{10}-h_{12}Y_{10}\end{pmatrix}\right]$$
$$+h_{14}Y_{10}\left[h_2S_{10}-\left(h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10}\right)\right]$$
$$\times\left[h_2S_{10}+h_{14}Y_{10}-\left(h_8-2h_9X_{10}+h_{10}S_{10}-h_{12}Y_{10}\right)\right]$$
$$+h_{12}h_{15}X_{10}Y_{10}\left[h_{14}Y_{10}-\begin{pmatrix}h_8-2h_9X_{10} \\ +h_{10}S_{10}-h_{12}Y_{10}\end{pmatrix}\right]$$

Here $\lambda_I^{[10]}$ denote to the eigenvalue in the $I-$direction. The Routh-Hurwitz conditions require $F_i>0\ \forall i=1,3$ and $\Delta=F_1F_2-F_3>0$, follows from condition (6b) and in addition the negativity of the other eigenvalues, namely condition (6a). So, according to Routh-Hurwitz criterion E_{10} is locally asymptotically stable.

<u>Theorem (7):</u>

If the following conditions hold

$$h_1<2h_2S_{11}+\left(h_2+h_3\right)I_{11}+h_4X_{11} \quad \text{...............(7a)}$$
$$\frac{h_6h_{10}}{h_9}<h_3<\frac{h_9h_{10}X_{11}}{h_{11}I_{11}} \quad \text{............................ (7b)}$$
$$h_9h_{14}X_{11}Y_{11}>h_3I_{11}\left(h_2S_{11}+h_{11}X_{11}+h_7\right) \quad \text{...(7c)}$$

Then, the equilibrium point E_{11} is a locally asymptotically stable.

***<u>Proof</u>*:**

The Jacobian matrix of the system (1) at E_{11} is given by $J_{11}=\left(\beta_{ij}^{[11]}\right)$ where:

$\beta_{11}^{[11]}=h_1-2h_2S_{11}-\left(h_2+h_3\right)I_{11}-h_4X_{11}\ ;$
$\beta_{12}^{[11]}=-\left(h_7+h_2S_{11}+h_6X_{11}\right);\ \beta_{13}^{[11]}=-h_4S_{11}\ ;$
$\beta_{14}^{[11]}=0\ ;\ \beta_{21}^{[11]}=h_3I_{11}\ ;\ \beta_{22}^{[11]}=0\ ,$
$\beta_{23}^{[11]}=-h_6I_{11}\ ;\ \beta_{24}^{[11]}=0\ ;\ \beta_{31}^{[11]}=h_{10}X_{11}\ ;$
$\beta_{32}^{[11]}=h_{11}X_{11}\ ;\ \beta_{33}^{[11]}=-h_9X_{11}\ ;\ \beta_{34}^{[11]}=-h_{12}X_{11}\ ;$
$\beta_{41}^{[11]}=\beta_{42}^{[11]}=0\ ;\ \beta_{43}^{[11]}=h_{15}Y_{11}\ ;\ \beta_{44}^{[11]}=-h_{14}Y_{11}\ .$

So, the characteristic equation of J_{11} can be written by

$$P\left(\lambda^{[11]}\right)=\left(\lambda^{[11]}\right)^4+F_1\left(\lambda^{[11]}\right)^3+F_2\left(\lambda^{[11]}\right)^2+F_3\left(\lambda^{[11]}\right)+F_4$$

with

$$
\begin{aligned}
F_1 &= -\left(\beta_{11}^{[11]} + \beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
F_2 &= \beta_{11}^{[11]}\beta_{44}^{[11]} + \beta_{33}^{[11]}\beta_{44}^{[11]} - \beta_{12}^{[11]}\beta_{21}^{[11]} - \beta_{23}^{[11]}\beta_{32}^{[11]} \\
&+ \beta_{11}^{[11]}\beta_{33}^{[11]} - \beta_{13}^{[11]}\beta_{31}^{[11]} - \beta_{34}^{[11]}\beta_{43}^{[11]} \\
F_3 &= \beta_{11}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]} + \beta_{44}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]} + \beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\
&- \beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]} - \beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]} - \beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]} \\
&+ \beta_{44}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]} + \beta_{11}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} + \beta_{33}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]} \\
F_4 &= \beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]} + \beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]}\beta_{44}^{[11]} \\
&+ \beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]}\beta_{44}^{[11]} - \beta_{11}^{[11]}\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\
&- \beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]}
\end{aligned}
$$

$$
\begin{aligned}
\Delta = F_1F_2 - F_3 &= \beta_{11}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]} - \left(\beta_{11}^{[11]}\right)^2\beta_{44}^{[11]} \\
&- 2\beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]} - \left(\beta_{11}^{[11]}\right)^2\beta_{33}^{[11]} + \beta_{11}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]} \\
&+ \beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]} - \beta_{11}^{[11]}\left(\beta_{33}^{[11]}\right)^2 - \left(\beta_{33}^{[11]}\right)^2\beta_{44}^{[11]} \\
&+ \beta_{44}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]} + \beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} + \beta_{44}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]} \\
&- \beta_{11}^{[11]}\left(\beta_{44}^{[11]}\right)^2 - \beta_{33}^{[11]}\left(\beta_{44}^{[11]}\right)^2 \\
&+ \beta_{13}^{[11]}\left(\beta_{32}^{[11]}\beta_{21}^{[11]} + \beta_{33}^{[11]}\beta_{31}^{[11]}\right)
\end{aligned}
$$

$$
\begin{aligned}
\Delta F_3 - (F_1)^2 F_4 &= \left(\beta_{11}^{[11]}\right)^2\beta_{34}^{[11]}\beta_{43}^{[11]} \\
&\times\left[\begin{array}{l}
\beta_{13}^{[11]}\beta_{31}^{[11]} - \beta_{11}^{[11]}\beta_{44}^{[11]} - \beta_{11}^{[11]}\beta_{33}^{[11]} \\
-2\beta_{33}^{[11]}\beta_{44}^{[11]} - \left(\beta_{33}^{[11]}\right)^2 - \left(\beta_{44}^{[11]}\right)^2
\end{array}\right] \\
&- \left(\beta_{11}^{[11]}\right)^2\beta_{33}^{[11]}\beta_{44}^{[11]}\left[\begin{array}{l}
2\beta_{13}^{[11]}\beta_{31}^{[11]} - \beta_{11}^{[11]}\beta_{44}^{[11]} \\
-\beta_{11}^{[11]}\beta_{33}^{[11]} - 2\beta_{33}^{[11]}\beta_{44}^{[11]} \\
-\left(\beta_{33}^{[11]}\right)^2 - \left(\beta_{44}^{[11]}\right)^2 \\
+2\beta_{12}^{[11]}\beta_{21}^{[11]} + 2\beta_{23}^{[11]}\beta_{32}^{[11]}
\end{array}\right] \\
&+ \left(\beta_{11}^{[11]}\right)^2\left[\begin{array}{l}
\left(\beta_{44}^{[11]}\right)^2\beta_{13}^{[11]}\beta_{31}^{[11]} - \left(\beta_{44}^{[11]}\right)^2\beta_{12}^{[11]}\beta_{21}^{[11]} \\
-\left(\beta_{33}^{[11]}\right)^2\beta_{23}^{[11]}\beta_{32}^{[11]} + \beta_{33}^{[11]}\beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]} \\
+\beta_{13}^{[11]}\beta_{31}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} + \beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\
-\beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}
\end{array}\right] \\
&+ \beta_{34}^{[11]}\beta_{43}^{[11]}\left[\begin{array}{l}
\beta_{11}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
+\beta_{44}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
+\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
+\beta_{11}^{[11]}\beta_{13}^{[11]}\left(\beta_{32}^{[11]}\beta_{21}^{[11]} + \beta_{33}^{[11]}\beta_{31}^{[11]}\right) \\
+\beta_{11}^{[11]}\beta_{23}^{[11]}\left(\beta_{33}^{[11]}\beta_{32}^{[11]} + \beta_{12}^{[11]}\beta_{31}^{[11]}\right) \\
+\beta_{11}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
-\beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
-\beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]}\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right)
\end{array}\right] \\
&+ \beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]}\left[\begin{array}{l}
\beta_{33}^{[11]}\beta_{44}^{[11]}\left(\beta_{33}^{[11]} + \beta_{33}^{[11]}\right) \\
-\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} - \beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\
+\beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]} - \beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\
-\beta_{44}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]} - \beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]}
\end{array}\right] \\
&+ \beta_{13}^{[11]}\beta_{32}^{[11]}\beta_{21}^{[11]}\left[\begin{array}{l}
\beta_{11}^{[11]}\left(\beta_{33}^{[11]}\right)^2 - \beta_{11}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]} \\
-\beta_{11}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]} - \beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}
\end{array}\right] \\
&+ \left(\beta_{33}^{[11]}\beta_{31}^{[11]} + \beta_{32}^{[11]}\beta_{21}^{[11]}\right)\left[\begin{array}{l}
\beta_{44}^{[11]}\beta_{31}^{[11]}\left(\beta_{13}^{[11]}\right)^2 \\
-\beta_{32}^{[11]}\beta_{21}^{[11]}\left(\beta_{13}^{[11]}\right)^2 \\
-\beta_{13}^{[11]}\left(\beta_{44}^{[11]}\right)^3 \\
-\beta_{33}^{[11]}\beta_{13}^{[11]}\left(\beta_{44}^{[11]}\right)^2 \\
+\beta_{44}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{13}^{[11]} \\
-\beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{13}^{[11]} \\
+\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\beta_{13}^{[11]} \\
-2\beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{13}^{[11]} \\
-\beta_{11}^{[11]}\beta_{13}^{[11]}\left(\beta_{44}^{[11]}\right)^2
\end{array}\right] \\
&- \beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\beta_{44}^{[11]}\beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]} \\
&- 2\beta_{11}^{[11]}\beta_{34}^{[11]}\beta_{43}^{[11]}\left(\beta_{33}^{[11]}\beta_{44}^{[11]} + \beta_{12}^{[11]}\beta_{21}^{[11]}\right)\left(\beta_{33}^{[11]} + \beta_{44}^{[11]}\right) \\
&+ \beta_{23}^{[11]}\beta_{32}^{[11]}\left[\left(\beta_{44}^{[11]}\right)^2\beta_{13}^{[11]}\beta_{31}^{[11]} + \beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\right]
\end{aligned}
$$

$$+\beta_{11}^{[11]}\beta_{44}^{[11]}\begin{bmatrix} \left(\beta_{13}^{[11]}\beta_{31}^{[11]}+\beta_{12}^{[11]}\beta_{21}^{[11]}\right)^2+\beta_{13}^{[11]}\beta_{31}^{[11]} \\ \times\beta_{23}^{[11]}\beta_{32}^{[11]}+\left(\beta_{44}^{[11]}\right)^2\beta_{13}^{[11]}\beta_{31}^{[11]} \\ -\left(\beta_{44}^{[11]}\right)^2\beta_{12}^{[11]}\beta_{21}^{[11]}-\beta_{44}^{[11]}\beta_{12}^{[11]} \\ \times\beta_{23}^{[11]}\beta_{31}^{[11]}+\beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]} \end{bmatrix}$$

$$+\beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]}\begin{pmatrix} \beta_{13}^{[11]}\beta_{31}^{[11]}-\beta_{33}^{[11]}\beta_{44}^{[11]} \\ +\beta_{23}^{[11]}\beta_{32}^{[11]}-\left(\beta_{44}^{[11]}\right)^2 \\ +\beta_{12}^{[11]}\beta_{21}^{[11]} \end{pmatrix}$$

$$+\beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{31}^{[11]}\begin{bmatrix} \beta_{44}^{[11]}\left(\beta_{13}^{[11]}\beta_{31}^{[11]}+\beta_{12}^{[11]}\beta_{21}^{[11]}\right) \\ -\beta_{21}^{[11]}\beta_{13}^{[11]}\beta_{32}^{[11]}-\left(\beta_{44}^{[11]}\right)^3 \\ +\beta_{23}^{[11]}\beta_{32}^{[11]}\left(\beta_{11}^{[11]}+\beta_{44}^{[11]}\right) \end{bmatrix}$$

$$+\left(\beta_{33}^{[11]}\beta_{21}^{[11]}-\beta_{23}^{[11]}\beta_{31}^{[11]}\right)\begin{bmatrix} \beta_{11}^{[11]}\beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{13}^{[11]} \\ +\beta_{11}^{[11]}\beta_{21}^{[11]}\left(\beta_{12}^{[11]}\right)^2 \\ -\beta_{12}^{[11]}\beta_{33}^{[11]}\left(\beta_{11}^{[11]}\right)^2 \\ -\beta_{11}^{[11]}\beta_{12}^{[11]}\left(\beta_{33}^{[11]}\right)^2 \\ +\beta_{33}^{[11]}\beta_{12}^{[11]}\beta_{23}^{[11]}\beta_{32}^{[11]} \\ -\beta_{11}^{[11]}\beta_{33}^{[11]}\beta_{44}^{[11]}\beta_{12}^{[11]} \\ +\beta_{23}^{[11]}\beta_{31}^{[11]}\left(\beta_{12}^{[11]}\right)^2 \\ +\beta_{12}^{[11]}\beta_{21}^{[11]}\beta_{13}^{[11]}\beta_{32}^{[11]} \\ +\beta_{33}^{[11]}\beta_{12}^{[11]}\beta_{13}^{[11]}\beta_{31}^{[11]} \end{bmatrix}$$

So, if $\beta_{11}^{[11]}<0\,;\,\beta_{33}^{[11]}\beta_{21}^{[11]}<\beta_{23}^{[11]}\beta_{31}^{[11]}$
$\beta_{33}^{[11]}\beta_{31}^{[11]}<-\beta_{32}^{[11]}\beta_{21}^{[11]}\,;\,\beta_{33}^{[11]}\beta_{44}^{[11]}>-\beta_{12}^{[11]}\beta_{21}^{[11]}$
and by using the Routh-Hurwitz conditions require $F_i>0\ \forall i=1,3,4,\Delta=F_1F_2-F_3>0$ and $(F_1F_2-F_3)F_3-(F_1)^2F_4>0$, which reduces to conditions (7a)-(7c). So, according to Routh-Hurwitz criterion E_{11} is locally asymptotically stable.

Numerical Simulations

To study the system (1) numerically let's use the cont. line (—) for x, dash line (- -) for y, dot line(::) for z and dash-dot line(-.-) for p in the all of the following figures. Now, consider the following set of parameters

$$\begin{aligned}&h_1=0.8, h_2=0.02, h_3=0.7, h_4=0.7, h_5=0.3,\\&h_6=0.6, h_7=0.18, h_8=0.9, h_9=0.03, h_{10}=0.6,\\&h_{11}=0.5, h_{12}=0.3, h_{13}=0.7, h_{14}=0.07,\\&h_{15}=0.2, q=0.6, E=0.5;\end{aligned} \quad ....(8)$$

With initial point $(0.75, 0.75, 0.75, 0.75)$. For this set of parameter (8), the solution trajectory of system (1) approaches to the equilibrium point
$E_{11}=(1.046\,,1.119\,,0.42\,,6.915)$ see Fig.(1).

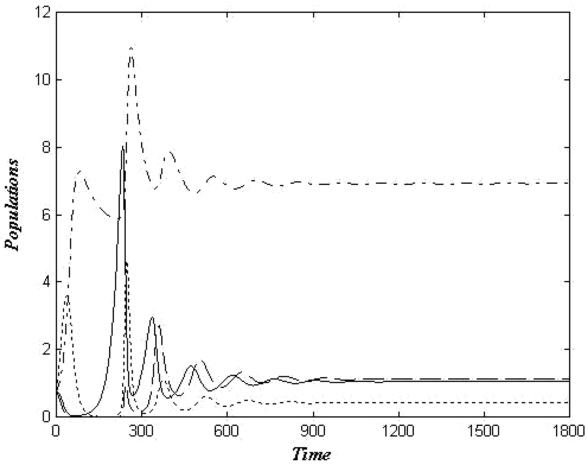

Fig.(1) Time series of the solution trajectory of system (1) for data given in Eq.(8) which show that E_{11} is a locally asymptotically stable.

If the infection rates constant $h_3\le 0.21$ then the solution trajectory of system (1) approaches to the equilibrium point $E_{10}=(2.923\,,0\,,1.059\,,8.741)$ see Fig.(2).

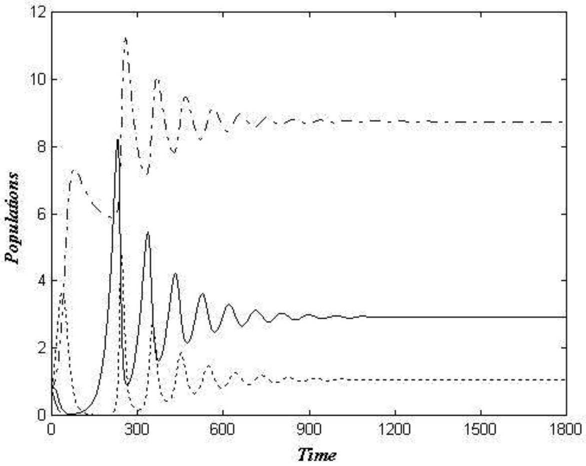

Fig.(2) Time series of the solution trajectory of system (1) for data given in Eq.(8) with $h_3=0.21$.

But, when the effect of catch-rate qE is considered then the solution trajectory of system (1) approaches to the equilibrium point $E_7=(0\,,0\,,1.993\,,2.801)$ when $q\ge 0.95, E\ge 0.95$ see Fig.(3), and approaches to the equilibrium point
$E_9=(0.686\,,2.783\,,0\,,9.714)$ when $q\le 0.2, E\le 0.1$ see Fig.(4).

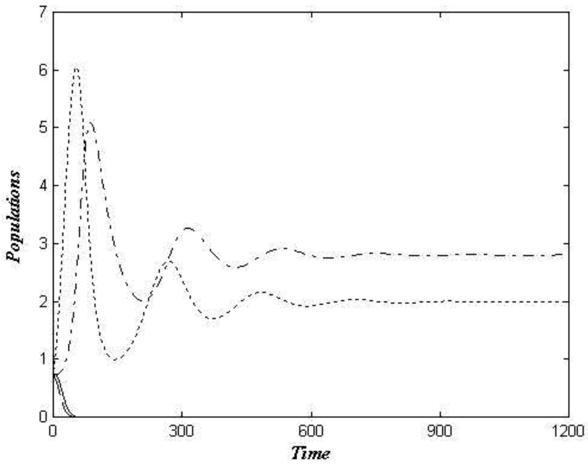

Fig.(3) Time series of the solution trajectory of system (1) for data given in Eq.(8) with $q = 0.95,\ E = 0.95$.

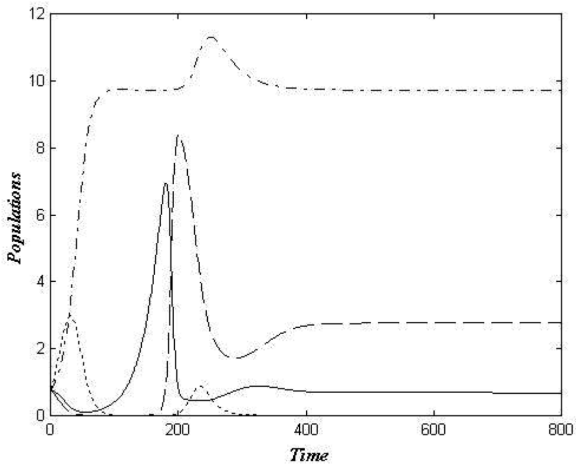

Fig.(4) Time series of the solution trajectory of system (1) for data given in Eq.(8) with $q = 0.2\,,\ E = 0.1$.

Conclusions and Discussion

In this paper, we proposed and analyzed the combined effect of SIS disease and harvest on a Food Chain model. The dynamical behaviour of system (1) has been investigated locally. In addition to assumed that the top predator population is harvested under optimal conditions, we used linear functional response and incidence rate for the diseases in prey species. The model included four non-linear autonomous differential equations that describe the dynamics of four different populations namely susceptible prey (S), infected prey (I), intermediate predator (X) and top predator (Y). The conditions for existence and stability for each equilibrium points are obtained. Similar, numerically explained that the solution trajectory of system (1) with parameters given in eq.(8) approaches to the equilibrium point $E_{11} = (0.891\,, 1.602\,, 0.24\,, 7.428)$. The system (1) is solved numerically for varying of infection rate h_3 keeping other parameters fixed as given in Eq.(8), then the solutions trajectory of system (1) are drawn in Fig.(2), it is clear that, as the infection rate decreases the infected individuals started decreases and the system (1) return to asymptotically stable at the coexistence equilibrium point in the Int. $\Re_{+}^{4}$. However, decreasing h_3 further, say $h_3 = 0.21$, causes losing the stability and then the solutions trajectory approaches to another equilibrium point E_{10}. The same way with parameters $q \geq 0.95\,; E \geq 0.95$, but with $q \geq 0.2\,; E \geq 0.45$ the solutions trajectory approaches to another equilibrium point E_9.

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الخلاصة

تصف هذه الورقة نموذج سلسلة غذائية ذو ثلاثة مجتمعات مع دالة استجابة خطية وحادثة إصابة بمرض. هذا النموذج يتضمن الفريسة، مفترس متوسط والمفترس الاعلى حيث يوجد حصاد في مجتمع المفترس الاعلى ومرض من نوع SIS ينتشر في مجتمع الفريسة مأخوذة بعين الاعتبار. تحليل الاستقرارية لجميع النقاط الثابتة للنظام درس. تم مناقشة تأثير كل من الحصاد والمرض على استقرار هذا النموذج. وأخيرا، استخدامنا المحاكاة العددية للتحقق من النتائج التحليلية.